UNCLASSIFIED

AD **257 982**

Reproduced
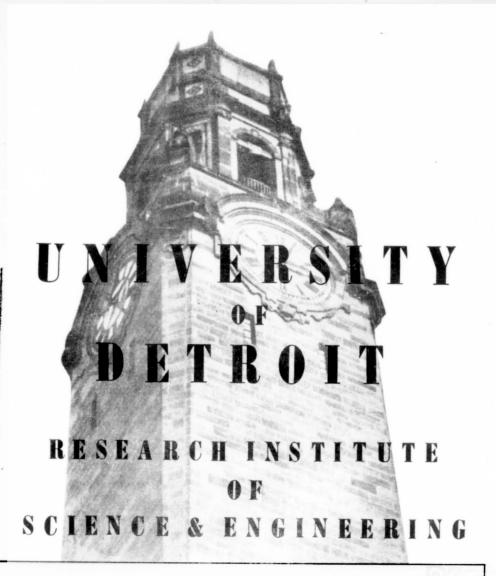*by the*

UNCLASSIFIED

UNIVERSITY OF DETROIT

RESEARCH INSTITUTE OF SCIENCE & ENGINEERING

INTERIM REPORT 1g

GRAVITATIONAL RADIATION IN FLAT SPACETIME RELATIVITY

Under Technical Supervision of

U.S.A. TRECOM   Contract No. DA-44-177-TC-519

Prepared by: James C. Keith   February 8, 1961.

4001 W. McNICHOLS ROAD   DETROIT 21, MICHIGAN

GRAVITATIONAL RADIATION

IN

FLAT SPACETIME RELATIVITY

INTERIM REPORT Ig

of

Work Done by the

UNIVERSITY OF DETROIT

Under the Technical Supervision of

U.S.A. TRECOM

Date of Report

February 8, 1961

Prepared on Contract No. DA-44-177-TC-519

By

James C. Keith

GRAVITATIONAL RADIATION IN FLAT SPACETIME RELATIVITY

James Clark Keith[*]

Research Institute, University of Detroit

ABSTRACT

The possibility of a $1/c^3$ order gravitational radiation is considered within the framework of the flat spacetime relativistic gravitational theory of G. D. Birkhoff. The method stresses the complementarity between results derived by considering direct interparticle actions and the intensity of the far gravitational radiation field. It is proved that centrifugal as well as gravitational forces may dampen the accelerated motion of a system of masses. The results predict that a "freely" rotating mass system will dissipate energy at a rate directly proportional to its total energy and to the factor $-k(a\omega/c)^3\omega$ 1/sec, ω being the angular velocity, 'a' the radius of rotation, c the velocity of light in vacuo, and k a dimensionless ratio of the system's moments.

For a small rotor magnetically suspended and spun in ultrahigh vacuum, the gravitational power loss can be detected directly by measuring the angular deceleration. An experimental decision should help shed light on the validity of the more generally accepted $1/c^5$ order radiation predicted by other means.

Purpose.

Using the typical flat spacetime relativistic gravitational theory of G. D. Birkhoff,[1] we shall apply the theory to prediction of possible gravitational effects of an order higher than $1/c^2$. Several effects of interest appear in connection with the associated gravitational field and radiation from accelerating masses.

One result of special interest is the prediction of a new, alternate type of 'pure' gravitational radiation which lies directly between the $1/c$ order gravitational aberration effect suggested by Laplace[2] and the $1/c^5$ order gravitational quadrupole radiations predicted by Einstein and others.[3] The new predictions appear susceptible to experimental verification. This is interesting in that _no_ gravitational phenomena of an order higher than $1/c^2$ have _ever_ been observed. In the last section an _experimentum_ _crucis_ is proposed to test the validity of the resulting predictions.

Notations needed.

$g_0^{ij} = g_{ij}^o = (1,-1,-1,-1)$ for $i = j$ $\qquad$ $\delta_{ij} = (1,1,1,1)$ for $i = j$
$\qquad\qquad\quad = 0$ for $i \neq j$ $\qquad\qquad\qquad\qquad = 0$ for $i \neq j$

$i,j,k,m,n = 1,2,3,4$ (spacetime components). $\qquad$ $\alpha = 2,3,4$ (space components only).

$\Box^2 = \vec{\Box} \cdot \vec{\Box} = g^{ij} \partial^2/\partial x^i \partial x^j$ where the standard summation convention holds on upper-lower repeated indices. $\qquad$ $\vec{\nabla}$ is the 3-vector del operator.

γ the gravitational constant. $\qquad$ $\gamma = 1/(1-[v/c]^2)^{\frac{1}{2}}$ the Lorentz factor.

h_{ij} gravitational potential tensor. B_{jk}^i gravitational field tensor.

T_{ij}, $t_{ij}^g (r,ct)$ stress-energy-momentum tensors of mass and gravitational field.

u^j 4-velocity component. $\qquad$ $\vec{v}$ 3-vector velocity.

$\vec{\alpha}$ 3-vector acceleration. $\qquad$ $\alpha^2 = \vec{\alpha} \cdot \vec{\alpha}$

f_j 4-force component. $\qquad$ $\vec{F}$ 3-vector force.

ω angular velocity (radians/sec). $\qquad$ $c \cong 3 \times 10^{10}$ cm/sec speed of light in vacuo.

Units.

Unrationalized c.g.s. when needed.

I. BASIS AND SOME INTERESTING CONCLUSIONS OF THE BIRKHOFF THEORY

I.A Postulates of the relativistic gravitational theory of G. D. Birkhoff.

The formal basis of the Birkhoff gravitational theory is embodied in the following postulates expounded by Birkhoff, Graef, and Barajas:[4]

B.1 The frame of reference of gravitational phenomena is the flat $(+,-,-,-)$ spacetime of Lorentz-Minkowski.

B.2 The formal principle of equivalence is assumed.[5]

B.3 The gravitational field is completely characterized in spacetime by a doubly covariant symmetrical potential tensor h_{ij}. The gravitational field components B^i_{jk} are linear combinations of the first partial derivatives of h_{ij}.

B.4 The h_{ij} of a mass point at rest with respect to an inertial system is equal in spacetime to the product of the Newtonian potential $\mathfrak{A}_0 = \underline{\gamma}M/r$ times the doubly covariant Kronecker delta.

B.5 Gravitational perturbations propagate with the velocity of light in vacuo in an inertial reference system.

B.6 The h_{ij} of a mass point in arbitrary motion is equal to the set of all the instantaneous gravitational fields generated by this mass point in all its positions in physical space.

B.7 The potential tensor of a gravitational field caused by many mass points is equal to the sum of the gravitational potential tensors of each mass point.

I should like to draw special attention to Birkhoff's crucial concept of the limiting velocity of "disturbance" in the perfect fluid of his theory.[6] In light of the axiomatic nature of this concept, B.5 might perhaps be worded more suggestively:

B.5 Inertial and gravitational disturbances, forces, and field potentials propagate at fundamental velocity c of magnitude equal to the speed of light in vacuo, independent of the mass or energy density or velocity of the frame of reference.

Some very pointed arguments of Moshinski[7] concerning the complementary pure gravitational field interactions can be summarized by adding two postulates to the above system. These give a concise description of the field aspects of Birkhoff Theory.

B.8 Each component of the potential tensor h_{ij} in rectangular orthogonal coordinates satisfies the inhomogeneous wave equation: $\Box^2 h_{ij} = 4\pi\gamma\rho_{ij}$ [8]

B.9 The field interaction energy density is given by the Lagrangian function,[7]

$$L = -1/8\pi\gamma \; g^{ij} \; \partial h^{mn}/\partial x^i \; \partial h_{mn}/\partial x^j + \tfrac{1}{2}h^{mn}\rho'_{mn}$$

where ρ'_{mn} is the equivalent mass density of the external field.

B.1 - B.7 embody the direct interparticle gravitational action description in much the same way the "new" electrodynamics of Moon and Spencer[9] gleaned from the 19th Century physics describes the direct interparticle action in electromagnetic theory. Postulates B.8 and B.9 manifest the field aspects of the gravitational theory proposed by Birkhoff. Both field and particle descriptions will be of interest.

The above postulates are consistent with Birkhoff's initial formulation (1943) and must be assumed to generate a completely self-contained flat spacetime relativistic theory of gravitational interaction. The theory as it stands is a strong sequel to the first attempts by Nordstrom (1912) and Whitehead (1922) to develop a relativistic gravitational theory within the uniform flat spacetime of Lorentz-Minkowski.

I.B Applications of the theory.

The gravitational theory of Birkhoff may be summed up by the equations:[1]

$$c^2 d^2 x_i/ds^2 = f_i^g/m = (\partial h_{ij}/\partial x^k - \partial h_{jk}/\partial x^i)u^j u^k$$

$$(1)$$

$$g^{ij} \; \partial^2 h_{mn}/\partial x^i \partial x^j = 4\pi\gamma\rho_{mn} \quad \text{or in free space, 0.}$$

$$(i,j,m,n \text{ run from 1 to 4})$$

The "true" mass m is assumed constant along the world tube "S"; $c^2 d^2 x_i/ds^2 = f_i^g/m$ is the 4-acceleration component of a particle acted on by the gravitational force, and measures the curvature of the world tube which that (extended) particle pervades. u^j are components of the world velocity of m and constitute the unit tangent vector to the world tube. g^{ij} is the metric tensor for a flat spacetime. h_{ij} are symmetric components of the gravitational tensor potential.

In a free stationary space surrounding and proper to a point mass M, the Poisson equation of (1) becomes time-independent and yields a proper static solution for h_{ij}:[1]

$$h_{ij} \begin{matrix}(\text{static}) \\ (\text{point mass})\end{matrix} = -\delta_{ij}\gamma M/r \qquad (2)$$

where δ_{ij} is the Kronecker delta, $= 1$ when $i=j$ and 0 when $i \neq j$. This is also Birk-hoff's solution for the static, spherically symmetric central mass where integration extends over the volume of "true" mass of the Birkhoff perfect fluid of density $\varphi_{ij} = \rho_o \delta_{ij}$. Only point-mass and spherically symmetric solutions of h_{ij} will be considered in this article unless further integration is indicated.

Graef has shown[10] that when the mass M moves with arbitrary velocity, the relativistic potential encountered in an inertial frame of a remote observer is obtained by ordinary transformation laws for a second order tensor in flat spacetime:

$$h_{ij} = (2u_i u_j - g_{ij}^o)h_o \qquad (3)$$

Here u_i, u_j are covariant components of _relative_ velocity between M of h_o and the rest frame of an inertial observer, h_o being the Newtonian potential $-\gamma M/r$. Graef also shows how more refined considerations will require all quantities to be taken at their retarded times with r in the denominator of h_o replaced by $[r - (\vec{r} \cdot \vec{v})/c]$ in usual manner.[10]

Substitution of the static h_{ij} of (2) into the force equations (1) gives the equations of motion for a small point mass m moving in the field of M with relative velocity components u^j, u^k. Birkhoff integrated the force equations and derived the correct crucial test predictions from his new theory in 1942. The theory so far has predicted accurately _all_ observed gravitational phenomena. The correct factor of 2 in the gravitational bending of light appears as a direct and non-arbitrary consequence of the Theory as opposed to such implications as drawn by W. Thirring[11]. In addition Graef has predicted the apsidal advance in the general problem of two bodies without excessive effort,[4] and Romero Juarez has done like-wise with a restricted three-body problem of great interest.[12]

In 1945, Alba realized that were a central mass M to rotate about its axis, its gravitational field would be modified somewhat as suggested by the relativistic dependence of the potential tensor in (6).[13] It was not until 1952 that Alba applied his result to the calculation of non-planar forces on the orbit of a satellite,[14] thus generalizing Birkhoff's equations for the relativistic motion of a point mass in a static field.

Alba's arguments seem satisfactory from a phenomenological point of view and consistent within the framework of the Birkhoff Theory. The axial rotation of an extended mass M perturbs its static field because of relation (3) derived by Graef. The static field of M in its proper frame must first be transformed into a frame rotating with minus the angular frequency of M. Thereby a new field of M is set up in all inertial spaces, in particular (to a good approximation) in that space in which a satellite orbits about M. Once this new field of M has been found by (3), the remaining relativistic corrections depend only on the velocity of the orbiting mass m with respect to the center of mass of M.

Spin acceleration thus appears to perturb the static central field in a singular and irremovable manner. Since Graef has shown that h_{ij} appears diagonalized only in its proper rest frame, it can be inferred that absolute motion or acceleration of a source M will in general always affect its external field in the non-accelerated "free" space posed by the universe, as will be shown more rigorously in a later section.

The external gravitational fields of accelerating masses are therefore distinguishable in the Birkhoff Theory in an absolute sense. The new external field potential of an accelerating mass has the character of a non-diagonalized tensor of second order. The complexity of the total external field is thus enhanced in a definitive way.

A natural extension of Alba's argument treats the effect of acceleration of the orbital mass m. Let v^j be components of 4-velocity of M relative to the center of mass of an m-M system, and u^j be the components of velocity of m also relative to the system's center of mass. Let us compare the results of substituting the potentials given by (3) for m into the force equations (1). The acceleration

experienced by M as a result of the force exerted on it by the field of m follows immediately:

$$c^2 d^2 x_1/ds^2 = (2u_1 u_j v^j - v_1) v^k \partial h_0/\partial x^k - (2u_j v^j u_k v^k - 1) \partial h_0/\partial x^1$$

$$+ 2h_0 v^j v^k (\partial u_i u_j/\partial x^k - \partial u_j u_k/\partial x^1) \tag{4}$$

The last term drops out only if the non-diagonal components of the tensor potential vanish. The acceleration experienced by m as a result of the force exerted on it by M obtains by permuting the roles of the v^j, u^j, and h_0 in (4).

To illustrate the physical significance of these equations, consider the ideal situation where m rotates in a circle about M so that h_0 and $\gamma = 1/[1 - (v/c)^2]^{\frac{1}{2}}$ are time-independent constants of the motion and the ratio m/M<<1. Under these conditions the 3-vector acceleration of M is found from equation (4) to be:

$$\vec{F}_g/M = -(2\gamma^2 - 1)\vec{\nabla} h_0 - 2\gamma^2 (2\gamma^2 - 1) h_0 \vec{a}_+/c^2 \tag{5}$$

where $h_0 = -\gamma m/r$ and $\vec{a}_+$ is m's acceleration in the positive $\hat{r}$ direction. The last term of (5) shows the gravitational force acts also on the equivalent mass of the m-M field energy. The factor 2 suggests the gravitational mass is twice the equivalent field energy mass, which appears to be localized about m. When m accelerates, the equivalent mass of its field interaction energy also accelerates. The first term on the right of (5) is the main relativistic correction term of the Birkhoff Theory. If the ratio of m/M is not vanishingly small, additional residual terms will appear in the description of the motion because of the velocity of M relative to the center of mass.

I.C. Question of theoretical and experimental comparisons between the gravitational theories of Einstein and Birkhoff.

It would be interesting to find reasonable in-range experimental tests for deciding definitely in favor of the curved or flat spacetime versions of gravitational theories. One wonders from the standpoint of the models involved how many residual levels of fine effects the Einstein and Birkhoff theories in fact describe.

Verification of $1/c^2$ order "crucial tests" such as advance of Mercury's peri-
nelion, bending of light around the sun, and shift of the emission line of Mössbauer
γ-rays in the Earth's gravitational and rotary centrifugal fields, brilliantly con-
firm the Birkhoff theoretical predictions as well as those of Einstein. At the same
time, the Mössbauer experiments check the locally indistinguishable nature of the ef-
fects of gravitational and centrifugal fields on electromagnetic waves as commonly
assumed in both theories. Such close theoretical correspondence makes one skeptical
that decisive tests to decide in favor of either theory will by found in the area
of $1/c^2$ effects. The theories down to this level appear equally valid representations
of nature.

This situation suggests the desirability of examining residual gravitational
effects of order higher than $1/c^2$. Do accelerating systems radiate away energy
through their gravitational field and run down?

There are yet no experiments which determine the existence, magnitude, or velo-
city of propagation of "pure" gravitational radiation (or even induction) fields.
The "crucial tests" above indicate very indirectly that the characteristic velocity
of gravity may well equal the velocity of light in vacuo.[7] According to Laplace, if
gravity propagates at some fundamental velocity, aberration of the central forces ex-
perienced by a satellite should cause $1/c$ order tangential forces to affect continu-
ously the satellite's secular equation.[2] Paradoxically, if c is the speed of light
in vacuo, such a first order effect would have been observed astronomically long ago.

On the other hand, the $1/c^5$ order Einstein gravitational radiation reaction
(which Eddington considers a residual, fifth order Laplace effect) remains beyond
the range of practical measurement.[15]

The direction taken by the previous considerations of Birkhoff, Graef, Moshinski,
and Alba have led me to apply the Birkhoff Theory for the first time to effects of an
order higher than $1/c^2$. In the absence of experimental results, such residual grav-
itational effects are only possible descriptions of the physical systems involved.
This effort is motivated in part by the fact that the ensuing predictions appear barely
to be within the range of experimental observation. If this were indeed the case, the
results offer a definitive experimental test for the respective models deriving from
Einstein and Birkhoff theories for the cases of gravitational radiation from acceler-
ating mass systems.

II. RADIATION REACTION ON MASS ACCELERATING IN GRAVITATIONAL FIELDS

II.A Time component of gravitational 4-force on a mass m.

To study in a simple way the gravitational radiation process, it would be well to formulate a measure of the total time rate of change of energy in a mass system. With only gravitational forces acting, the time component of inertial force according to the Birkhoff theory must equal the time component of gravitational force, i.e.

$$c f_1 = \gamma d/dt\,(\gamma m c^2) = c f_1^g \tag{6}$$

By imposing Birkhoff's condition of orthogonality of the world velocity and world force, the corresponding relation $u^i f_i = 0$ gives that

$$u^1 f_1^g = -(u^2 f_2^g + u^3 f_3^g + u^4 f_4^g) \tag{7}$$

or

$$c f_1^g = -c f_g^1 = -\vec{v} \cdot \vec{F}_g \tag{7a}$$

where $\vec{v}$ is the non-relativistic 3-vector velocity and $\vec{F}_g$ is the relativistic Birkhoff 3-vector force given by summing the spatial $(2,3,4)$ components of the gravitational 4-force defined by (1). Combining (6) and (7a) gives the net time rate of change of energy of the mass m in an external gravitational field:

$$\gamma d/dt\,(\gamma m c^2) = \vec{v} \cdot \vec{F}_g = dE/dt \tag{8}$$

The average of this quantity over a long time or integral number of revolutions T is defined by the time-averaging operation:

$$\overline{dE/dt} \quad 1/T \int_0^T \vec{v} \cdot \vec{F}_g\, dt \tag{9}$$

With a long term or integral cycle gravitational radiation loss, the time component of gravitational force would not average to zero and would affect m's inertial energy. Since the gravitational induction field is usually considered the conservative part of the total field, its fluctuations average to zero in this process.

Since the Birkhoff Theory is intended to describe <u>completely</u> all gravitational interactions, and the conservative induction field components vanish,* we assume what is left from equation (9) contains information about the gravitational radiation or radiation-reaction undergone by an accelerating system of masses. The time-averaging operation on dE/dt in (8) is implicitly carried out in our following results.

II.B Subsidiary field conditions.

Some refinement is necessary to account for the finite velocity of propagation of fields. Such subsidiary conditions derive naturally from the postulates of the first section. Graef derives retardation conditions for the gravitational potential tensor using postulates B.5-B.8.[10] Except that they apply to a second order tensor, these conditions are the same as those of the standard Lienard-Wiechert retarded potentials of electromagnetic theory so familiar in the literature. Consistent results can be obtained usually by expansion of the fields and velocities measured at the origin of the observer in a Taylor Series about the point $t_0 - r_i/c$, where r_i is the distance to each source-mass element dm_i. Physically, this allows the field at the center of mass of a system at time t_0 to be expressed by superposing contributions from each dm_i at time r_i/c earlier.

It can be shown that the net effect of the retardation conditions on the source moving with uniform velocity is to make the total force appear to come from the <u>present</u> rather than the retarded position of the source.[16] When additionally the source accelerates, residual tangential components can arise in the force field and cause a degradation of the energy of an interacting system of masses. We examine this situation in detail in the next sections.

II.C Field retardation.

In Part I we have seen how acceleration of a mass according to the Birkhoff Theory modifies its static gravitational field. When a force acts on a mass causing it to

---

* Tidal frictions and the like are not considered in the present study.

accelerate, it also acts to accelerate the negative equivalent mass of the gravitational field energy. The result is summed up by equation (5), which holds down to the $1/c^2$ order of residual gravitational effects.

Graef has shown, as pointed out in the last section, that h_{ij} must actually be treated as a retarded tensor potential. As mentioned, the net effect on the Newtonian potential h_0 is to make the gradient of h_0 at the origin appear to come from the present position of the source to a very good approximation. The modified tensor potential however also involves derivatives of the retarded velocity components of m. When expanded out in a Taylor series about the point $t_0 - r/c$, r being the distance to the center of mass of the system,

$$(\vec{v})_{t-r/c} = \vec{v}_0 - (r/c)d\vec{v}_0/dt + (r^2/2!c^2)d^2\vec{v}_0/dt^2 \ldots \ldots \ldots \ldots \quad (10)$$

where $\vec{v}_0$ is the present vector velocity. We restrict ourselves to physical systems where v possesses continuous, well behaved derivatives, and $v < c$. We shall consider only terms down to the first order in $1/c$. The last term of equation (5) now becomes:

$$\ddot{\vec{F}}_g/m = -2\gamma^2(2\gamma^2 - 1)h_0\ddot{\vec{a}}_0/c^2 + 2\gamma^2(2\gamma^2 - 1)(h_0 r/c^3)d^2\vec{v}_0/dt^2 \quad (11)$$

The rate of energy dissipation from the last term is then:

$$\vec{v} \cdot \ddot{\vec{F}}_g = +2\gamma^2(2\gamma^2 - 1)(mh_0 r/c^3)\vec{v}_0 \cdot d^2\vec{v}_0/dt^2 \quad (12)$$

The minus sign indicates the system dissipates power when $+mh_0 = -\gamma Mm/R$. M can be any mass at distance R (at rest relative to the origin) which interacts with m.

II.D Radiation reaction in various physical systems of interest.

The result lends itself to an interesting physical interpretation: The gravitational force between m and M would be almost central except that the equivalent mass of their mutual field energy, which is localized about m, lags m in space because of the above retardation condition on the velocity components. The direct gravitational force between that (negative) equivalent mass of the field energy and M then tries to pull m "backwards" and opposes the motion.

In the ideal case of a mass m rotating in a circle with h_0 = a constant and with v^2 an invariant of the motion, the velocity is perpendicular to the acceleration and:

$$\vec{v} \cdot \vec{F}_g = -2\gamma^2 (2\gamma^2 - 1)(\gamma Mm/r)(r\omega/c)^3 \omega \qquad (13)$$

This is proportional to the square of the absolute acceleration and apparently has no angular dependence. Since the time average of the square of the acceleration is the square of the acceleration, the result for a ring of mass orbiting around a central mass M becomes

$$(\vec{v} \cdot \vec{F}_g)_{ring} = -2\gamma^2 (2\gamma^2 - 1)(\gamma Mm_{ring}/r)(r\omega/c)^3 \omega \qquad (14)$$

whereas if the ring of mass holds itself together gravitationally and spins at the same time around its center of mass,

$$\vec{v} \cdot \vec{F}_g = -2\gamma^2 (2\gamma^2 - 1)\gamma m^2 (r\omega^2)^2/c^3 \qquad (15)$$

for an equal, balancing mass pole rotating in opposition to m, both $\vec{v}$ and $\ddot{\vec{v}}$ are reversed and the power losses of the combined masses are additive.

Equation (13) gives the energy dissipated in a unit time from an M-m planetary system. Evidently, the energy is lost continuously and at a constant rate. The ratio of the power dissipated to the total negative energy of the M-m system is positive, so that the kinetic energy continues to increase and the planet spirals in towards M. For the Earth-Sun system, letting T be the present period of 1 year,

$$dT/dt \cong -2(r\omega/c)^3 \omega T \qquad (16)$$

Since all the values are known, immediate substitution and integration of the effect over 10^8 years results in the prediction that the year will be 10 hours shorter than now due to the gravitational reaction derived above. The present long term stability of atomic clocks falls just slightly short of the accuracy required for the measurement of such an effect. Both m and M should radiate a power due to their own absolute accelerations. If r and a are the distances of m and M from the center of mass

of the m-M system, $\underline{mr} = \underline{Ma}$; $\vec{F}_g^m \cdot \vec{v}_m$ is found to equal $r_g^{\vec{M}} \cdot \vec{v}_M$; and the sum equals:

$$-4\gamma^2 (2\gamma^2 - 1) \underline{\gamma m^2} (r\omega^2)^2/c^3 \qquad \qquad (17)$$

The power dissipated by the mutual interaction of m and M is therefore M/m times the power either mass dissipates individually.[*] As the ratio of M/m approaches 1 as in the case of the rotating dumbbell, the power dissipated in the mutual inter-action approaches that dissipated by the masses individually.

To calculate the self interaction of an accelerating mass, let the center of mass of m be at $Q(r,t)$ while $P(r', t_0 - (r' - r)/c)$ is a non-accelerated point of an inertial space "instantaneously" occupied by m. The field at P originating at Q is retarded by a time $t' = r/c$. Considering the interaction of dm' with the static field of m, we find the approximate power dissipated in the self-interaction by integrating dm' over the entire mass distribution. This gives a net resistance to m's (linear) acceleration of:

$$\vec{F}_g = -\gamma^2 (2\gamma^2 - 1) \underline{\gamma m^2} (d^2 \vec{v}_0/dt^2) 1/c^3 \qquad \qquad (18)$$

The power dissipated by the m-m dumbbell will be compatible with this result if we assume coherence will make the final result $N^2 = 4$ times the power from one of the masses.

[*] See equation (13).

III. THE FAR FIELD

III.A How does acceleration effect the energy density of the far gravitational field?

In accordance with the findings of Part I, only components of gravitational field in the direction of absolute acceleration of source mass m will be relevant to first order calculation of the far field gravitational radiation intensity. This appears reasonable on grounds that field energy density has equivalent mass which reacts in line with the source's apparent acceleration as shown in equation (5).

The Lagrangian function Moshinski chooses (Postulate B.9) for the energy density of a Birkhoff gravitational field reduces in free space in absence of other fields to:

$$L = (-1/8) \pi_{\underline{\gamma}} g^{ij} (\partial h^{mn}/\partial x^i)(\partial h_{mn}/\partial x^j) \tag{19}$$

In this section we concern ourselves with the way the first order radial component h_{1r} of h_{ij} changes along r and contributes to outward flow of gravitational radiation from the accelerating mass system. Thus we write

$$L = (-1/8) \pi_{\underline{\gamma}} g^{ij}(\partial h^{1r}/\partial x^i)(\partial h_{1r}/\partial x^j) = (-1/4)\pi_{\underline{\gamma}}(\vec{\nabla} h_{1r})^2 \tag{20}$$

where $h_{1r} = -h^{1r} = -2\gamma^2 h_0 v_r/c$, and we use this to get the self-interaction of the far field.

The energy density in the far field by symmetry must propagate outward in the radial direction with finite velocity c (Postulate B.5). In time dt, an amount of energy $dE = L \, dV = Lc \, dt \, dA$ then flows out through the differential area dA. The total time rate of energy change within the volume enclosing the accelerating mass system is:

$$dE/dt = \oint_A L\vec{c} \cdot d\vec{A} \tag{21}$$

Since $dA = r^2 d\Omega$ where Ω is the solid angle subtended by dA at distance from the origin r, only terms in the field energy density (20) which go as $1/r^2$ will have no singularities at infinity and will indicate the net gravitational radiation flow dependent only on source parameters. We therefore exclude terms of higher order than $1/r^2$.

This exclusion will be indicated by the new symbol $\overset{\bullet}{\underset{0}{=}}$ which signifies that only $1/r$ dependent terms of the field and $1/r^2$ dependent terms of the field energy density L are to be considered.

III.B Linearly and circularly accelerating mass. (Diagrams 1-2)

Let m, moving from rest at $0(0,0,0,0)$ with acceleration $\vec{a}$ along the x-axis, be at a point $Q(t,x,0,0)$ while $P(t',x',y',z')$ is a non-accelerated point of "absolute" space at which the field energy density is to be found. From classical mechanics, $\frac{1}{2}mv^2 = max$ holds between the velocity, acceleration, and x coordinate of m at any time t. Thus $\vec{\nabla}v_x = a/v$. Since the y and z components of velocity are zero, (3) gives:

$$h_{1r} = -2\gamma^2 h_0 (v_x/c)\cos(r,v) \tag{22}$$

where $\cos(r,v)$ is the angle between OP and QP. The gradient of h_{1r} is

$$\vec{\nabla}h_{1r} \underset{0}{=} (-2\gamma^2 h_0/c)\cos(r,v)\ \alpha/v \tag{23}$$

and equation (20) immediately gives L. Using this in conjunction with equation (21), we get finally:

$$dE/dt \underset{0}{=} -\oint_A (d\Omega/4\pi)(4\gamma^4 \underline{\gamma}m^2/c)(\alpha/v)^2\cos^2(r,v) = (-4/3\gamma^4)(\underline{\gamma}m^2/c)(\alpha/v)^2 \tag{24}$$

where if ϕ is the angle between the projection of r on the x-y plane and the x-axis, and θ the angle between the r and the z-axis, $d\Omega\cos^2(r,v) = \sin^3\theta\ \cos^2\phi\ d\theta\ d\phi$.

Because overall momentum must be conserved for physical systems, the picture of an accelerating monopole of mass is too ideal. In reality, $m\vec{a}_1 = M\vec{a}_2$ where M moves in the opposite direction to counterbalance the motion of m. We now take m equal to M and find how this modifies the total far field potential.

The velocities of the two mass points m and m are now equal and opposite, and by Postulate B.7 the total potential at a far field point P is:

$$H_{1r} = -2\gamma^2(h_0/c)(v_x^{(1)} - v_x^{(2)})\cos(r,v) \tag{25}$$

So far retardation conditions on the field interactions have been neglected, in which case the above potential would vanish. A little thought. however shows the potential of the nearer mass to be slightly out of phase with that from the further mass due to the finite field propagation velocity c. The difference in phase of the two masses as seen at P is just $2(x/c)\cos(r,v)$. A previous relation (10) then gives to the first order:

$$v_x^{(2)} = v_x^{(1)} - 2(x/c)\cos(r,v)\alpha_x^{(1)} \tag{26}$$

The total potential therefore does not vanish, and its gradient is approximately:

$$\vec{\nabla}H_{1r} = -4\gamma^2(h_0\vec{\alpha}/c^2)\cos^2(r,v) \tag{27}$$

It follows immediately that:

$$dE/dt = \oint_A L\vec{c} \cdot d\vec{A} = (-4/5)\gamma^4 \underline{\gamma}(2m)^2\alpha^2/c^3 \tag{28}$$

An equivalent line of reasoning gives similar results to the above for the case of circularly accelerating mass. Again, calculation of h_{1r} requires that the velocity of the circling mass be projected onto the radius vector to the far field point P. The angular dependence of the h_{1r} field becomes $\sin\theta\sin(\phi - \omega t)$ where ω is the angular velocity of rotation and θ and ϕ retain the same meaning as above. For the case of the rotating dumbbell, retardation conditions are accounted for in the phase difference in the $\sin(\phi - \omega t)$ term. The total potential at P then becomes:

$$H_{1r} = -4\gamma^2(h_0\vec{x}\cdot\vec{\alpha}/c^2)\sin^2\theta\,\cos^2(\phi - \omega t) \tag{29}$$

The total power dissipated can then be gotten as above by computing the gradient of H_{1r} and from it L, then integrating L over the appropriate enclosing area. This of course gives back exactly the result in equation (28).

There is another point of even more interest in the expression for H_{1r} in this case of circularly accelerating mass. Namely if H_{1r} represents contributions from opposite mass elements of a rotating ring of mass, integration over all the mass in-

volves the term $\cos^2(\phi - \omega t)d\Omega$ integrated from 0 to π. But this integral does not vanish, and therefore H_{1r} at a far field point although becoming time independent in this case does not vanish. Neither does its gradient or the outward flowing gravitational field energy density L! Integration of L over the enclosed area containing the rotating mass ring then gives a steady flow of energy out of the system:

$$dE/dt \underset{\tilde{o}}{=} (-8/15)\gamma^4 \underline{\gamma} M^2 \alpha^2/c^3 \qquad (30)$$

Here M is the total mass of the ring.

Equations (28) and (30) should be compared with (17) and (15) for the radiation reaction for a mass dumbbell and ring derived in Part II. The agreement is fairly close, but not exact numerically. The shortcoming may derive from not considering other components of h_{1j} in the second order computation involving the retardation conditions, or it may be irremovable in the same way that the electromagnetic mass of the electron does not account for its entire mass.

There is an interesting physical interpretation of what happens when m accelerates. In the Birkhoff Theory acceleration of a source mass has been seen to modify its gravitational field. One can imagine the field acquires a new energy density which propagates out from the mass at c. One might look on the induction field action also as propagating out at c in the static field case. However the net energy "flowing out" of such a field by (20) goes as $1/r^2$ and thus rapidly drops to zero with increasing r. The net energy flow from the residual accelerated part of the field has no r dependence. It therefore gives a net energy loss at distances far from the system.

This type of gravitational radiation cannot be thought of in terms of oscillating multipole fields. It appears solely a result of the physical acceleration of an interacting mass system.

III.C Coherence.

When N masses make up each pole of a spinning dumbbell, (28) tells us the power radiated is N^2 times the single m-m dipole. This is the case for radiation from N coherent sources.[17] For most macroscopic, physically realistic systems, the wavelength

$\lambda = 2\pi c/\omega_{system}$ is much greater than the physical dimensions. The coherence is a result of residual radiation components emitted in phase with each other.

Next, one might naturally ask what is the gravitational radiation loss for a Nm-m dipole spinning about its center of mass. In this case m undergoes an acceleration α_0 and Nm an acceleration $\alpha_0 1/N$ about the center of mass. (28) shows each pole radiates proportional to its gravitational field energy and the square of its cwn acceleration. The two masses taken with their respective accelerations are thus found to radiate equally. Due to coherence, the net power is just 4 times that expected from the single accelerating pole alone.

So far the mutual gravitational field energy of m and Nm has not been taken into account. Assuming the mutual field energy is localized about m for reasons explained in Parts I and II, the accelerational interaction for large N dissipates a power equal to N times that dissipated by the single accelerating pole alone. In general an "amplitude" can be defined proportional to the square root of the gravitational field energy and to the absolute value of the total acceleration. Because of coherence the power is the square of the total amplitude, but for large N the cross terms and the separate contributions from m and Nm can be neglected with respect to the radiation term caused by the mutual gravitational energy. Equivalently, the single pole m can by thought of as interacting separately with each one of the N masses of the opposite pole, each pair contributing incoherently to a net power again approximately N times that from the single accelerating pole alone.

An m-M gravitational system can therefore lose energy into its far field at a rate proportional to the mutual gravitational potential energy of the system. Such an interpretation is consistent with the estimates of radiation reaction found in Part II. The radiation reaction dissipation (13) has therefore been correlated with the far field gravitational radiation in the framework of the Birkhoff Theory. As mentioned previously, as m approaches the size of M, i.e. m = M, the two interpretations definitely become equivalent, and the total power radiated goes as the mutual potential interaction energy of the masses.

IV. TWO PICTURES OF GRAVITATIONAL RADIATION LOSS

IV.A Integration of the preceding views of radiation from accelerating masses.

It may prove helpful to integrate the particular examples of the preceding sections into a general view of gravitational radiation in flat spacetime. The perfect fluid[1] of Birkhoff, though purely an auxiliary concept and not essential to the Birkhoff Theory, offers a formal connection between the gravitational and inertial forces and will be used here to aid in linking the complementary pictures of direct interparticle radiation reaction forces and the far gravitational radiation field intensity.

The equation of state of the Birkhoff perfect fluid, whose internal mass-energy and momentum is characterized by the tensor $T^{ij} = T^{ji}$ and gravitational field by $t_g^{ij}(r,ct)$ at each point, is defined by $p = \rho_{oo}c^2$ where the pressure and density are scalar world invariants.

T^{ij} describes solely the state of the "true" mass. It therefore vanishes everywhere outside the world tube of a mass particle m. $t_g^{ij}(r,ct)$ describes all complementary gravitational field effects. With only gravitational forces acting, the divergence of the sum of T^{ij} and $t_g^{ij}(r,ct)$ must vanish to ensure equilibrium of forces. When the proper rest-energy density as here is assumed a priori constant along the world line of m, the total energy density generally can fluctuate between kinetic and gravitational. In absence of gravitational fields, this provides the standard continuity relations for the perfect fluid, while in absence of "true" masses we obtain the conservative character of the static gravitational field and continuity of any radiation fields which may occur (although equivalent mass in the field may also be a source of radiation).

Integrating the divergence of $T^{ij} + t_g^{ij}(r,ct)$ over space, and applying the theorem of Gauss in 4 dimensions, $\oint_A [T^{i\alpha}]_A dA = 0$ since $T^{i\alpha} = 0$ outside the world tube and:

$$\int_V \gamma\frac{\partial T^{11}}{\partial t}\,dV_o = -\int_V \gamma\frac{\partial}{\partial t}\,t_g^{11}(\vec{r},ct)\,dV - \oint_A \{ct_g^{1\alpha}(\vec{r},ct)\}_A dA_\alpha \qquad (31)$$

$$\alpha = 2,3,4$$

By direct integration, the left term is found to be the time component of inertial Minkowski 4-force $(\gamma d/dt)\gamma mc^2$, the time rate of change of energy in the mass system.

IV.B• Radiation Reaction Picture.

The time rate of change of total inertial energy can be seen to be equal to the time component of Birkhoff gravitational 4-force multiplied by c. The sum of the last two terms of (31) is then by definition just $mc\,B^1_{jk}\,u^ju^k$. This represents the rate at which external gravitational forces acting on m do work. Because of the conservative character of the gravitational induction field, this therefore expresses the residual gravitational radiation reaction when averaged over a full cycle or long time interval. Note that $mc\,B^1_{jk}\,u^ju^k$ depends only on relative parameters, i.e. mutual energy, relative velocity and acceleration, etc., all in a framework of flat spacetime. Thus we have a direct interparticle calculation of radiation damping or friction as opposed to the complementary picture which handles gravitational radiation solely on a basis of field interactions.

IV.C Direct Field Interaction Picture.

Besides calculating directly the self or external field reaction of each mass element dm and summing over all elements to obtain the total time rate of change of energy, a far field radiation intensity, which should yield a compatible result, can be obtained by a complementary method. Let us say the energy of a gravitational radiation field at distance r from m propagates in the direction of r with fundamental velocity c. The energy crossing an area dA in time dt will be found at far distances in a volume c dA dt. Using Moshinski's field energy density, Lagrangian function, the total power dissipated over an enclosing surface will be:

$$dE/dt = \oint_A L\vec{c}\cdot d\vec{A} = -\oint_A \{(\vec{c}/8\pi\gamma)g^{ij}(\partial h^{mn}/\partial x^i)\partial h_{mn}/\partial x^j\}\cdot d\vec{A} \tag{32}$$

This result can be compared with the selection of the Lagrangian energy density for the electromagnetic field and with the corresponding form of the electromagnetic Poynting vector.

The first term on the right in relation (31) vanishes for our purposes because, a) the time average of the conservative induction field is independent of time, and b) the local energy density of the gravitational radiation field itself can be assumed in the case of "steady" radiation processes to be constant on the time average. The last term describes the total flow of gravitational radiation energy through a bounding surface and will be taken to be independent of the distance r from the center of the radiating mass system. Under these conditions, the quantity in brackets, $\{ct_g^{1\alpha}(\vec{r},ct)\}$, compares with the quantity in brackets in the previous relation (32), and both of these take the form of a gravitational Poynting-type-vector describing the radiation flow, a function in general of the distance r and the polar angles θ and $\phi_0 - \omega t$.

The dual aspect of our radiation picture is now apparent. In the case of the far field, the angular and $1/r^2$ dependence of the gravitational radiation intensity will enter through relation (32), whereas in the direct interparticle interaction of mass m with self or external gravitational forces, no necessity appears for evoking the angular dependence of the emitted radiation. Resolution of this possibly significant difference in our radiation pictures would require more scrutiny than applied here, and remains a problem of certain interest. A point in favor of these pictures of gravitational radiation is that both otherwise seem completely consistent with the Birkhoff Theory, which indeed was originally intended to describe completely all gravitational effects. One must only be aware that more refinements, such as the retardation conditions imposed by Graef,[10] are required in the treatment of effects of higher than $1/c^2$ order.

V. REACTION OF CENTRIFUGAL FORCES ON FREELY SPINNING MASSES.

V.A Interaction of a freely spinning mass with the universe.

Interaction of the gravitational field of an accelerating mass element m in a distribution of elements whose center of mass remains at rest with respect to a large remote external mass M can be calculated directly from equation (4). Noting the only non-vanishing component of v_M^j in this case is $v^1 = 1$, and if $\vec{F}_g$ is the relativistic 3-vector force which m exerts on M at a distance R so great that $1/R^2$ and higher order terms can be neglected, the last term of (4) becomes:

$$\vec{F}_g \overset{=}{\underset{0}{}} 2\gamma^4 [1 + (v/c)^2] (\underline{\gamma Mm/R}) \vec{a}_+/c^2 \tag{33}$$

Here $\vec{v}$ denotes the velocity of the element m relative to its system's center of mass (and hence in this case to M).

It may be well before going on to indicate the physical meaning of (33). Previous arguments of Graef[*] suggest that in the Birkhoff Theory the limiting value of $\underline{\gamma Mm/R}$ (as for example for the universe) is $\frac{1}{2}mc^2$. Were this indeed the case,

$$\vec{F}_g \overset{=}{\underset{0}{}} \gamma^4 [1 + (v/c)^2] m \vec{a}_+ \tag{34}$$

that is, there is a gravitational interaction with the universe approximately equal to the inertial (centrifugal) force on m. Indeed, this expresses Mach's contention that the so-called inertial force is actually of gravitational origin. The mutual gravitational field energy of m with respect to M again appears to be localized about m and to undergo the same acceleration.

Going on, we now ask what residual effects may occur in the interaction of the mass elements of a freely spinning mass with the universe. Because the acceleration of an element m must be referred to the center of spin, by equation (10) it must be treated as a retarded quantity. This results in the Taylor Series expansion analogous to equation (11) which gives a residual component of interaction force in line with the vector velocity of m. Neglecting all force reaction effects on the rest energy mc^2 of m, which has been assumed constant along the world line, and using

---

[*] "Orbit Theory", Proc. Symp. App. Math. Vol.IX, p.171, (1957)

the results of equations (9), (11), and (33), the kinetic energy of each mass element is found to decrease at a rate:

$$dK.E./dt = \vec{F}_g \cdot \vec{v} = -4\gamma^4 \, (K.E./c^2)\,(r/c)\vec{v} \cdot d^2\vec{v}/dt^2 \qquad (35)$$

where r is the spin radius of m and K. E. its kinetic energy. Since the product of $\vec{v}$ and its second time derivative is the same sign and a colinear product for an isolated, freely spinning system, the losses from each mass element are apparently additive and contribute to a net energy decay from the entire system. The "freely" spinning mass therefore will slow down due to its interaction with all masses in the universe.

This is strictly a gravitational type interaction. Physically, the net power dissipation in this case appears to be non-oscillatory and independent of direction. If the universe by definition cannot itself spin, the entire angular momentum re-action force must set back on the freely spinning mass system causing a real slowing down.

V.B Self interaction of mass elements in a freely spinning system.

Equation (34) suggests that centrifugal forces can be interpreted in the Birk-hoff Theory in terms of direct gravitational interaction with the universe. These forces according to postulate B.5 propagate at finite velocity c. One can therefore ask what residual effects arise in interaction of oppositely paired elements in a freely spinning mass distribution.

Consider Diagram 4. We ask whether the line of force between oppositely paired mass elements passes through the center of mass. If it did, it could be assumed no tangential components of force were acting on either mass. However we shall show how acceleration enters the picture to preclude the trivial result.

It takes a time $2a/c$ in the proper rest frame of the spinning system for a force disturbance from one mass element to propagate to the other. This has led or perhaps misled some authors[2,18] to assume the existence of $1/c$ order tangential forces within the system. However it is not hard to see that in the proper frame the oppositely paired elements are at rest with respect to each other and no such components actu-ally arise. However the propagation of centrifugal forces is invariant only with

respect to velocity in a flat spacetime, not to acceleration or its rates of change, as evidenced by the bending of light in a gravitational field. With respect to the propagating centrifugal forces which are not acceleration invariant, the opposite mass element in time $2a/c$ accelerates a distance in the tangential $\hat{\theta}$ direction of $1/6\dot{\alpha}(2a/c)^3$. The angular discrepancy which must appear in the centrifugal force is therefore $4/3(\dot{\alpha}a^2/c^3)$ so that a $1/c^3$ order component of that force does lie in the $\hat{\theta}$ direction. When this component is dotted with the velocity and the relativistic factors taken into account by transforming the process out of the proper frame of the spinning mass, the result for circular motion is just $2/3$ of the result found in equation (35). This suggests the two treatments, V.A and V.B, are very nearly equivalent.

Physically, in the rest frame of the spinning system the centrifugal forces would appear to diminish with time. This describes a new process which can be called the centrifugal damping of spin motion.

For the spinning point dipole of radius 'a', equation (35) gives:

$$dE/dt = 2\vec{F}_g \cdot \vec{v} = -4\gamma^4 mv^2 (a\omega/c)^3 \omega \tag{36}$$

and the proportional decay of kinetic energy for low velocities is:

$$\frac{d(K.E.)/dt}{K.E.} = + \frac{2d\omega/dt}{\omega} \simeq -4(a\omega/c)^3 \omega \tag{37}$$

It is particularly interesting that spontaneous loss of rotational kinetic energy, similar to the losses predicted from the gravitational systems in Parts II and III, does not appear to cancel when integrated over a mass such as a rotating sphere or cylinder whose external multipole moments do not vary with time. Since the "gravitational radiation" reaction in the case of the universal interaction is proportional directly to m, the contributions of each mass element to the total power dissipated are additive and equation (36) can be generalized to:

$$(dE/dt)_{\text{general radiation reaction}} = -\int_m 2\gamma^4 (r\psi/c)^3 r^2 \omega^3 \, dm \tag{38}$$

This expression sets a lower bound on the rate "freely" spinning mass systems slow down.

A lucid interpretation attaches to the separate situations of purely gravitational and centrifugal radiation damping. Physically, in (13) picture v/c components of a negative outward-flowing "g-field" fluid acting against the negative equivalent mass of the m-M field. In (35) picture v/c components of positive outward-flowing "centrifugal field" fluid to act against the positive equivalent mass of the kinetic energy. A little reflection will show both effects go in the same direction, causing a net rate of energy loss from the accelerating mass system in agreement with the minus signs which appear in these formulae.

Gravitational or/and centrifugal damping of a freely spinning mass system's motion therefore results in a positive outward flow of $1/c^3$ order gravitational radiation energy, accompanied by a genuine decrease in the total energy of the source system, i.e. a genuine decay of the motion. This is in contradistinction to the net effect of $1/c^5$ order gravitational quadrupole radiations predicted by Einstein and others,[3] whereby a negative energy mysteriously appears to be radiated outward while the total internal energy of the mass system increases. (See, for example, A. Peres, Il Nuovo Cimento, XI, 1 March 1959, p. 653).

V.C Experimental magnitude of the predicted $1/c^3$ order gravitational radiation.

For physically realistic, laboratory-sized, symmetrically rotating masses, acceptance of equation (38) leads to a proportional decay of angular frequency for freely spinning masses of magnitude:

$$\frac{d\omega/dt}{\omega} = (-2\gamma^4\omega^4/c^3) \int_m r^5 dm / \int_m r^2 dm \simeq -k(a/c)^3\omega^4 \tag{39}$$

k depends on the geometrical moments of the mass, being 8/7 for a right cylinder rotating about its axis, 1 for a thin rod rotating about an axis through its center normal to its length. The minus sign indicates that the spin frequency decreases.

With a view to experimental feasibility, for a steel cylinder spinning near the limits of tensile strength at velocity $a\omega \simeq 10^5$ cm/sec, the frequency decay ratio depends on ω^4. Let $a \simeq 10^{-3}$ cm with ω on the order of 10^8 radians/sec. $d\omega/dt$ is then found to be on the order of $\frac{1}{2}$ radians/sec/sec. The effect is thus on the order of 5 parts in 10^9 on frequency.

It seems within possibility simultaneously to generate and measure such quantities using a type of magnetic suspension and rotation apparatus developed by Beams.[19] By suspending the rotating mass in ultra-high vacuum and making use of suspension symmetry, virtually all stray frictions are impressively reduced. Moreover, magnetic and other residual drags appear to vary proportional to much lower powers of the frequency.[20a]

The angular frequency and its rates of change are capable of being detected with great precision by comparison with a good frequency standard. The experimental techniques appear hard but straightforward.[20b] On the other hand, tests for the gravitational quadrupole radiation of Einstein[3], even if reinterpreted in the sense of sections V.A and V.B, are of an $(a\omega/c)^2$ lower magnitude and far beyond the range of present instrumentation.
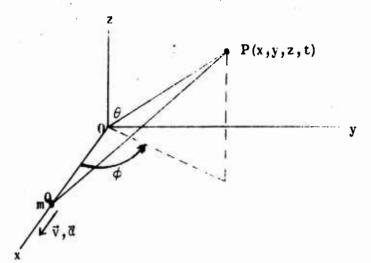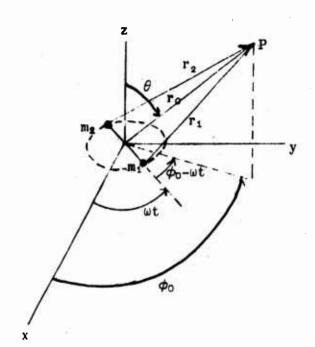
Diagram 1. Linearly accelerating mass.

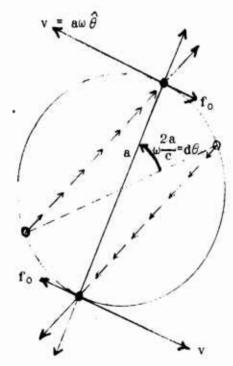Diagram 2 & 3. Circularly accelerating mass (mass dumbell).

Diagram 4a. Showing why v/c components of centrifugal force might be expected to retard the m-m dipole spin.

$$d\theta = 0 = vt/a; \quad v = 0$$
$$\delta\theta = \frac{1}{6}\omega^3 (2a/c)^3$$
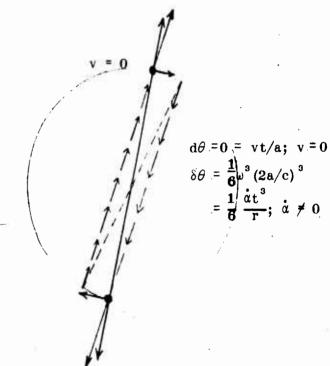$$= \frac{1}{6}\frac{\dot{\alpha}t^3}{r}; \quad \dot{\alpha} \neq 0$$

Diagram 4b. Showing that in the proper frame of the m-m dipole, the situation portrayed in 4a does not occur. However, while the centrifugal forces propagate at constant velocity, the mass system accelerates, causing a smaller but finite angular discrepancy and tangential force components.

REFERENCES

1. G. D. Birkhoff, Proc. Nat. Acad. Sc., 29, 8 pp.231-39 (1943).

    Bol. de la Soc. Mat. Mex., V.1, 4-5 pp.1-23 (1944).

    M. S. Vallarta, Encyclopedia Britannica "Gravitation", 10, pp.665-7.

    J. C. Keith, "On Birkhoff's Theory of Matter and Force" M.I.T. BS Thesis (1957).

    For other typical flat spacetime gravitational theories, see, for example:

    F. Belinfante & J. Swihart, Annals of Physics,1, pp. 168-212,281-99 (1957).

2. Pierre Simon Laplace, "Mechanique Celeste" X, vii, Sections 18-22 (1825).

3. Albert Einstein, Berlin Sitzungsberichte, p.688 (1916); p.154 (1917);

    A. Einstein & N. Rosen, J. Franklin Institute, 223, 43 (1937).

    For other examples, see proceedings of "Colloque International Sur Les Theories

    Relativistes de la Gravitation" Royaumont, France, 21-27 Juin (1959).

4. C. Graef Fernandez, Cosmos y Ciencia, 1, 1 pp.44-65 (1954).

5. A. Barajas, Proc. Nat. Acad. Sci., 30, 54 (1944).

    Soc. Mat. Mex., pp.51-56 Julio (1945).

6. G. D. Birkhoff, Proc. Nat. Acad. Sc., V.13, 3 pp.161-169 (1927).

7. M. Moshinski, Phys. Rev., 80, 4 pp.514-19 (1950).

8. See reference 1.

9. P. Moon & D. Spencer, J. Franklin Inst., V 257, pp.369-82 (1954).

10. C. Graef Fernandez, Rev. Mex. Fisica, V.1, 1 pp.11-27 (1952).

11. W. Thirring, Fortschritte der Physik, 7, p.100 (1959).

12. A. Romero Juarez, Bol. de la Soc. Mat. Mex., pp.13-19 (1946).

13. Fernando Alba, Bol. de la Soc. Mat. Mex., pp.57-63 Julio (1945).

14. Fernando Alba, Rev. Mex. Fisica, V.1, 1 pp.38-43 (1952).

15. A. S. Eddington, "Math. Theory of Rel." Sec. 57 & Note 8, Camb. Univ. Press (1924).

16. Panofsky & Phillips, "Class. Elec. & Mag." XIX Addison-Wesley (1955).

17. Slater & Frank, "Electromagnetism" XII, Sec. 6 McGraw-Hill (1947).

18. John Wimpffen, Babson Gravity Foundation Essay (1959).

19. J. W. Beams, Young, Moore, J. App. Phys., 17, pp.886-90 Nov. (1946).

20a. D. J. Kenney, J. Indus. Math. Soc., pp. 1-13 July (1960).

20b. J. Nixon, Interim Report VIa. Contract No. DA-44-177-TC-519 Sept. (1960).

DISTRIBUTION LIST

Address	No. of Copies
Commanding Officer U. S. A. TRECOM Fort Eustis, Virginia Attention: R.C.O.	6
Research Institute of Science and Engineering University of Detroit Detroit 21, Michigan	4
ASTIA Arlington Hall Arlington, Virginia	10
Dr. J. W. Beams Department of Physics University of Virginia Charlottesville, Virginia	1
Dr. William Bender Western Washington College Bellingham, Washington	1
Mr. Daniel Menzel Agricultural Experiment Station University of California Berkeley 4, California	1
Mr. I. W. Rozian Steel Division Ford Motor Company Dearborn, Michigan	1
Professor C. Graef Fernandez International Atomic Energy Agency Vienna, Austria	1
Dr. Garrett Birkhoff Department of Mathematics Harvard University Cambridge, Massachusetts	1
Dr. Alberto Barajas Facutad des Ciencias Universidad Nacional A. de Mexico Mexico 21, D. F.	1

Dr. Stanley Deser   1
Brandeis University
Waltham, Massachusetts

Professor J. L. Synge   1
Institute of Advanced Studies
Dublin, Ireland

Mr. N. Rosen   1
Institute of Technology
Haiffa, Israel